Notes on Mankiw, Romer and Weil
Allin Cottrell, September 2003

Here’s an exegesis of Section I of the paper by Mankiw, Romer and Weil (MRW). If you can get a good understanding of this section, the rest of the paper should be fairly easy.

On p. 409, MRW say, “We assume a Cobb-Douglas production function,” and their first equation is, accordingly,

\[ Y(t) = K(t)\alpha (A(t) L(t))^{1-\alpha} \quad 0 < \alpha < 1 \] (1)

A couple of points to note here:

• The repeated \((t)\) terms do not indicate multiplication by \(t\), they are just indicating that output, \(Y\), capital input, \(K\), labor input, \(L\), and the state of technology as it impacts the productivity of labour, \(A\), are all considered to be functions of time: they are all potentially changing over time.

• The exponents \(\alpha\) and \(1 - \alpha\) are both positive fractions, and they (obviously) add up to 1.

This is maybe a slightly more complicated-looking version of something you have seen in micro. Let’s simplify the production function a little (suppressing the \(t\)s):

\[ Y = AK^\alpha L^\beta \]

What’s the marginal product of labor (MPL) here? Basic calculus gives us

\[ \frac{dY}{dL} = AK^\alpha \frac{d(L^\beta)}{dL} = AK^\alpha \beta L^{\beta-1} = \beta \frac{AK^\alpha L^\beta}{L} = \beta Y \]

The ratio \(Y/L\) is output per worker, also known as the average product of labor or APL. Note that if \(\beta\) is a positive fraction, the MPL is less than the APL. In that case, the APL will fall as \(L\) rises (it is being pulled down at the margin). And since the MPL is just a constant, \(\beta\), times the APL, it too will fall as \(L\) rises. So with \(0 < \beta < 1\) we have positive but diminishing returns to labor. A parallel argument shows that \(0 < \alpha < 1\) is the condition for positive but diminishing returns to capital.

Now in the MRW version of the equation, they in effect put \(\beta = 1 - \alpha\). What is the significance of that? Well, imagine scaling both inputs, capital and labor, by a common factor \(\lambda\) (e.g. \(\lambda = 2\) would indicate a doubling of the inputs). What happens to the output? We can write the original output as

\[ Y_0 = AK_0^\alpha L_0^\beta \]

and the new output as

\[ Y_1 = A(\lambda K_0)^\alpha (\lambda L_0)^\beta = A\lambda^\alpha K_0^\alpha \lambda^\beta L_0^\beta = \lambda^\alpha \lambda^\beta AK_0^\alpha L_0^\beta = \lambda^{\alpha + \beta} Y_0 \]

and the ratio \(Y_1/Y_0\) is then just \(\lambda^{\alpha + \beta}\). In that case:

• If \(\alpha + \beta = 1\) a scaling of the inputs by \(\lambda\) produces an expansion of output by that same factor—constant returns to scale;

• if \(\alpha + \beta > 1\) a scaling of the inputs by \(\lambda\) produces a greater than proportional increase in output—increasing returns to scale or economies of scale; and

• if \(\alpha + \beta < 1\) we have decreasing returns to scale.
MRW’s writing of the capital and labour exponents as $\alpha$ and $1-\alpha$ therefore corresponds to the assumption of constant returns to scale.

Returning to the text, MRW next write down growth equations for labor and the level of technology:

\[ L(t) = L(0)e^{nt} \]  
\[ A(t) = A(0)e^{gt} \]

Here, $L(0)$ and $A(0)$ are just initial values for these variables. The idea is that they grow (or shrink) at continuously compounded rates $n$ and $g$ respectively. The magic number $e$ (=2.718, the base of natural logarithms) achieves the continuous compounding. This number has the cool property that \(\frac{d}{dx} e^x = e^x\).

The table and graph below illustrate how $L$ and $A$ would behave over time according to these equations, with $n = 0.03$, $g = 0.05$ and $L(0) = A(0) = 100$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$L(t)$</th>
<th>$A(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100.000</td>
<td>100.000</td>
</tr>
<tr>
<td>1</td>
<td>103.045</td>
<td>105.127</td>
</tr>
<tr>
<td>2</td>
<td>106.184</td>
<td>110.517</td>
</tr>
<tr>
<td>3</td>
<td>109.417</td>
<td>116.183</td>
</tr>
<tr>
<td>10</td>
<td>134.986</td>
<td>164.872</td>
</tr>
<tr>
<td>18</td>
<td>171.601</td>
<td>245.960</td>
</tr>
</tbody>
</table>

The next step in the argument is passed over fairly quickly. The authors note the assumption that a constant fraction of output, $s$, is invested; they define $k \equiv K/AL$ and $y \equiv Y/AL$; and they then tell us that the evolution of $k$ over time is given by

\[ \dot{k}(t) = sy(t) - (n + g + \delta)k(t) \]  

Let’s pause on this. First, a small point of notation: the dot indicates a time-derivative; for any variable $x$, $\dot{x} = dx/dt$. Since $k = K/AL$ we can write

\[ \dot{k} = \frac{1}{AL} \frac{dK}{dt} - \frac{K}{(AL)^2} \left[ \frac{dA}{dt} L + \frac{dL}{dt} A \right] \]

We want the derivative of $k$ with respect to time. We’ll have to use the quotient rule (for $K(t)/(A(t)L(t))$) and the product rule (for the $A(t)L(t)$ in the denominator). From this point I’m going to drop the explicit “(t)”s to make the notation less cluttered, but of course we have to remember that $K$, $A$ and $L$ are all functions of time.

By the quotient rule,

\[ \dot{k} = \frac{1}{AL} \frac{dK}{dt} - \frac{K}{(AL)^2} \left[ \frac{dA}{dt} L + \frac{dL}{dt} A \right] \]

and by the product rule,

\[ \dot{k} = \frac{1}{AL} \frac{dK}{dt} - \frac{K}{AL} \left[ \frac{A}{AL} \frac{dL}{dt} + \frac{L}{AL} \frac{dA}{dt} \right] \]

\[ \dot{k} = \frac{1}{AL} \frac{dK}{dt} - \left[ \frac{1}{L} \frac{dL}{dt} + \frac{1}{A} \frac{dA}{dt} \right] \frac{K}{AL} \]

Now consider $dK/dt$, the time-derivative of the capital stock. Since a fraction $s$ of output $Y$ is invested (i.e., used to augment the capital stock), the growth of $K$ would equal $sY$, if it were not for depreciation.
Depreciation (wear and tear, obsolescence) is working to reduce $K$ over time. The authors assume this effect is proportional to the size of the capital stock, with factor of proportionality $\delta$. Thus

$$\frac{dK}{dt} = sY - \delta K$$

Substituting this into the last equation above gives

$$\dot{k} = \frac{1}{AL} (sY - \delta K) - \left[ \frac{1}{L} \frac{dL}{dt} + \frac{1}{A} \frac{dA}{dt} \right] \frac{K}{AL}$$

$$\dot{k} = s \frac{Y}{AL} - \delta \frac{K}{AL} - \left[ \frac{1}{L} \frac{dL}{dt} + \frac{1}{A} \frac{dA}{dt} \right] K$$

Now cash in $y = Y/AL$ and $k = K/AL$ to get

$$\dot{k} = sy - \left[ \frac{1}{L} \frac{dL}{dt} + \frac{1}{A} \frac{dA}{dt} + \delta \right] k$$

Last point: $dL/dt$ is the simple time-derivative of $L$. The proportional rate of change of $L$ over time is this time-derivative divided by $L$. But equation (2) implies that this proportional rate is just $n$. That is,

$$\frac{1}{L} \frac{dL}{dt} = n$$

and by the same token

$$\frac{1}{A} \frac{dA}{dt} = g$$

So, finally,

$$\dot{k} = sy - (n + g + \delta) k$$

which is MRW’s equation (4).

MRW give a variant form of this equation where $y$ is replaced by $k^\alpha$:

$$\dot{k} = sk^\alpha - (n + g + \delta) k$$

It’s easy enough to see where this come from. First, from the production function (1),

$$Y = K^\alpha (AL)^{1-\alpha}$$

But since $y = Y/AL$,

$$y = \frac{K^\alpha (AL)^{1-\alpha}}{AL} = K^\alpha (AL)^{-\alpha} = \left( \frac{K}{AL} \right)^\alpha = k^\alpha$$